DEPTH OF FACTORS OF SQUARE FREE MONOMIAL IDEALS

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Abstract. Let I be an ideal of a polynomial algebra over a field, generated by r-square free monomials of degree d. If r is bigger (or equal) than the number of square free monomials of I of degree $d+1$ then $\operatorname{depth}_S I = d$. Let $J \subset I$, $J \neq 0$ be generated by square free monomials of degree $\geq d+1$. If r is bigger than the number of square free monomials of $I \setminus J$ of degree $d+1$ then $\operatorname{depth}_S I/J = d$. In particular Stanley's Conjecture holds in both cases.

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Introduction

Let $S = K[x_1, \ldots, x_n]$ be the polynomial algebra in n-variables over a field K, d a positive integer and $I \supset J$, $I \neq J$ be two square free monomial ideals of S such that I is generated in degree $\geq d$, respectively J in degree $d+1$. Let $\rho_d(I)$ be the number of all square free monomials of degree d of I. It is easy to note (see Lemma 1.1) that $\operatorname{depth}_S I/J \geq d$. Our Theorem 2.3 gives sufficient conditions which imply $\operatorname{depth}_S I/J = d$, namely this happens when

$$\rho_d(I) > \rho_{d+1}(I) - \rho_{d+1}(J).$$

Suppose that this condition holds. Then the Stanley depth of I/J (see [9], [1], or here Remark 2.7) is d and if Stanley's Conjecture holds then $\operatorname{depth}_S I/J \leq d$, that is the missing inequality. Thus to test Stanley's Conjecture means to test the equality $\operatorname{depth}_S I/J = d$, which is much easier since there exist very good algorithms to compute $\operatorname{depth}_S I/J$ but not so good to compute the Stanley depth of I/J. After a lot of examples computed with the computer system SINGULAR we understood that a result as Theorem 2.3 is believable. Since the application of the Depth Lemma gives in many cases only inequalities, we had to find for the proof special short exact sequences, where this lemma gives a precise value of $\operatorname{depth}_S I/J$.

The proof of Theorem 2.3 was found looking to many useful examples, two of them being presented here as Examples 2.1, 2.2. The above condition is not necessary to have $\operatorname{depth}_S I/J = d$ as shows Example 2.5. Necessary and sufficient conditions could be possible found classifying some posets (see Remark 2.6) but this is not the subject of this paper. If I is generated by more (or equal) square free monomials of degree d than $\binom{n}{d+1}$, or more general than $\rho_{d+1}(I)$, then $\operatorname{depth}_S I = d$ as shows

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our Corollary 3.4 extending [8, Corollary 3], which was the starting point of our research, the proof there being much easier. Remark 3.5 says that the condition of Corollary 3.4 is tight.

1. Factors of square free monomial ideals

Let $J \subset I \subset S$, $J \neq I$ be two nonzero square free monomial ideals and d a positive integer. Let $\rho_d(I)$ be the number of all square free monomials of degree d of I. Suppose that I is generated by square free monomials $f_1, \ldots, f_r$, $r > 0$ of degree $\geq d$ and J is generated by square free monomials of degree $\geq d+1$. Set $s := \rho_{d+1}(I) - \rho_{d+1}(J)$ and let $b_1, \ldots, b_s$ be the square free monomials of $I \setminus J$ of degree $d+1$.

Lemma 1.1. $\operatorname{depth}_S I,\ \operatorname{depth}_S I/J \geq d$.

Proof. By an argument of J. Herzog (see [8, Remark 1.2]) we have $\operatorname{depth}_S I \geq d$, $\operatorname{depth}_S J \geq d+1$. The conclusion follows applying the Depth Lemma in the exact sequence $0 \to J \to I \to I/J \to 0$. □

Lemma 1.2. *Suppose that $J = E + F$, $F \not\subset E$, where E, F are ideals generated by square free monomials of degree $d+1$, respectively $> d+1$. Then* $\operatorname{depth}_S I/J = d$ *if and only if* $\operatorname{depth}_S I/E = d$.

Proof. We may suppose that in E there exist no monomial generator of F. In the exact sequence

$$0 \to J/E \to I/E \to I/J \to 0$$

we see that the first end is isomorphic with $F/(F \cap E)$ and has depth $\geq d+2$ by Lemma 1.1. Applying the Depth Lemma we are done. □

Before trying to extend the above lemma is useful to see the next example.

Example 1.3. Let $n = 4$, $d = 1$, $I = (x_2)$, $E = (x_2x_4)$, $F = (x_1x_2x_3)$. Then $\operatorname{depth}_S I/E = 3$ and $\operatorname{depth}_S I/(E+F) = 2$.

Lemma 1.4. *Let H be an ideal generated by square free monomials of degree $d+1$. Then* $\operatorname{depth}_S I/J = d$ *if and only if* $\operatorname{depth}_S (I+H)/J = d$.

Proof. By induction on the number of the generators of H it is enough to consider the case $H = (u)$ for some square free monomial $u \notin I$ of degree $d+1$. In the exact sequence

$$0 \to I/J \to (I+(u))/J \to (I+(u))/I \to 0$$

we see that the last term is isomorphic with $(u)/I \cap (u)$ and has depth $\geq d+1$ by Lemma 1.1, since $I \cap (u)$ has only monomials of degree $> d+1$. Using the Depth Lemma the first term has depth d if and only if the middle has depth d, which is enough. □

Using Lemmas 1.2, 1.4 we may suppose always in our considerations that I, J are generated in degree d, respectively $d+1$, in particular f_i have degrees d.

Lemma 1.5. *Let $e \leq r$ be a positive integer and $I' = (f_1, \ldots, f_e)$, $J' = J \cap I'$. If* $\operatorname{depth}_S I'/J' = d$ *then* $\operatorname{depth}_S I/J = d$.

Proof. In the exact sequence

$$0 \to I'/J' \to I/J \to I/(I'+J) \to 0$$

the right end has depth $\geq d$ by Lemma 1.1 because

$$I/(I'+J) \cong (f_{e+1},\dots,f_r)/(J + (I' \cap (f_{e+1},\dots,f_r)))$$

and $I' \cap (f_{e+1},\dots,f_r)$ is generated by monomials of degree $> d$. If the left end has depth d then the middle has the same depth by the Depth Lemma. $\square$

Lemma 1.6. *Suppose that there exists $i \in [r]$ such that f_i has in J all square free multiples of degree $d+1$. Then* $\operatorname{depth}_S I/J = d$.

Proof. We may suppose $i = 1$. By our hypothesis $J : f_1$ is generated by $(n-d)$-variables. If $r = 1$ then the depth of $I/J \cong S/(J : f_1)$ is d. If $r > 1$ apply the above lemma for $e = 1$. $\square$

Lemma 1.7. *Suppose that $r \geq 2$ and the least common multiple $b = [f_1, f_2]$ has degree $d+1$ and it is the only monomial of degree $d+1$ which is in $(f_1, f_2) \setminus J$. Then* $\operatorname{depth}_S I/J = d$.

Proof. Apply induction on $r \geq 2$. Suppose that $r = 2$. By hypothesis the greatest common divisor $u = (f_1, f_2)$ have degree $d-1$ and after renumbering the variables we may suppose that $f_i = x_i u$ for $i = 1, 2$. By hypothesis the square free multiples of f_1, f_2 by variables x_i, $i > 2$ belongs to J. Thus we see that I/J is a module over a polynomial ring in $(d+1)$-variables and we get $\operatorname{depth}_S I/J \leq d$ since I/J it is not free. Now it is enough to apply Lemma 1.1. If $r > 2$ then apply Lemma 1.5 for $e = 2$. $\square$

Proposition 1.8. *Suppose that $r > s$ and for each $i \in [r]$ there exists at most one $j \in [s]$ with $f_i | b_j$. Then* $\operatorname{depth}_S I/J = d$.

Proof. If there exists $i \in [r]$ such that f_i has in J all square free multiples of degree $d+1$, then we apply Lemma 1.6. Otherwise, each f_i has a square free multiple of degree $d+1$ which is not in J. By hypothesis, there exist $i, j \in [r]$, $i \neq j$ such that f_i, f_j have the same multiple b of degree $d+1$ in $I \setminus J$. Now apply the above lemma. $\square$

Corollary 1.9. *Suppose that $r > s \leq 1$. Then* $\operatorname{depth}_S I/J = d$.

Proposition 1.10. *Suppose that $r > s = 2$. Then* $\operatorname{depth}_S I/J = d$.

Proof. Using Lemma 1.5 for $e = 3$ we reduce to the case $r = 3$. By Lemma 1.6 we may suppose that each f_i divides b_1, or b_2. By Proposition 1.8 we may suppose that $f_1 | b_1$, $f_1 | b_2$, that is f_1 is the greatest common divisor (b_1, b_2). Assume that $f_2 | b_1$. If $f_2 | b_2$ then we get $f_2 = (b_1, b_2) = f_1$, which is false. Similarly, if $f_3 | b_1$ then $f_3 \not| b_2$ and we may apply Lemma 1.7 to f_2, f_3. Thus we reduce to the case when $f_3 | b_2$ and $f_3 \not| b_1$. We may suppose that $b_1 = x_1 f_1$, $b_2 = x_2 f_1$ and x_1, x_2 do not divide f_1 because b_i are square free. It follows that $b_1 = x_i f_2$, $b_2 = x_j f_3$ for some $i, j > 2$ with $x_i, x_j | f_1$.

Case $i = j$

Then we may suppose $i = j = 3$ and $f_1 = x_3u$ for a square free monomial u of degree $d-1$. It follows that $f_2 = x_2u$, $f_3 = x_1u$. Let S' be the polynomial subring of S in the variables x_1, x_2, x_3 and those dividing u. Then for each variable $x_k \notin S'$ we have $f_ix_k \in J$ and so $I/J \cong I'/J'$, where $I' = I \cap S'$, $J' = J \cap S'$. Changing from I, J, S to I', J', S' we may suppose that $n = d+2$ and $u = \Pi_{i>3}^n x_i$. Then $I/J \cong (I : u)/(J : u) \cong (x_1, x_2, x_3)S/((x_1x_2) + L)S$, where L is an ideal generated in $T := K[x_1, x_2, x_3]$ by square free monomials of degree > 2. Then $\operatorname{depth}_S I/J = d - 1 + \operatorname{depth}_T(x_1, x_2, x_3)T/(x_1x_2, L)$. By Lemma 1.2 it is enough to see that $\operatorname{depth}_T(x_1, x_2, x_3)T/(x_1x_2)T = 1$.

Case $i \neq j$

Then we may suppose $i = 3$, $j = 4$ and $f_1 = x_3x_4v$ for a square free monomial v of degree $d-2$. It follows that $f_2 = x_1f_1/x_3 = x_1x_4v$, $f_3 = x_2f_1/x_4 = x_2x_3v$. Let S'' be the polynomial subring of S in the variables x_1, x_2, x_3, x_4 and those dividing v. As above $I/J \cong I''/J''$, where $I'' = I \cap S''$, $J'' = J \cap S''$. Changing from I, J, S to I'', J'', S'' we may suppose that $n = d+2$ and $v = \Pi_{i>4}^n x_i$. Then

$$I/J \cong (I : v)/(J : v) \cong (x_1x_4, x_2x_3, x_3x_4)S/((x_1x_2x_3, x_1x_2x_4) + L')S,$$

where L' is an ideal generated in $T' := K[x_1, x_2, x_3, x_4]$ by square free monomials of degree > 3. Then

$$\operatorname{depth}_S I/J = d - 2 + \operatorname{depth}_{T'}(x_1x_4, x_2x_3, x_3x_4)T'/((x_1x_2x_3, x_1x_2x_4) + L')T'.$$

By Lemma 1.2 it is enough to see that

$$\operatorname{depth}_{T'}(x_1x_4, x_2x_3, x_3x_4)T'/(x_1x_2x_3, x_1x_2x_4)T' = 2.$$

$\square$

Proposition 1.11. *Suppose that* $d = 1$ *and* $r > s$. *Then* $\operatorname{depth}_S I/J = 1$.

Proof. We may suppose that $I = (x_1, \ldots, x_r)$. If $r = n$ then $\operatorname{depth}_s S/I = 0$ and it follows $\operatorname{depth}_S I/J = 1$ by the Depth Lemma because $\operatorname{depth}_S S/J \geq 1$ by Lemma 1.1. Suppose that $r < n$. Using Lemma 1.6 we may suppose that each x_i, $i \in [r]$ divides a certain b_k. Apply induction on s, the case $s \leq 2$ being done in the above proposition. We may assume that $s > 2$ and $b_1 = x_1x_2$. If there exist no b_k, $k > 1$ in (x_1, x_2) then we may take $I' = (x_1, x_2)$, $J' = J \cap I'$ and we have $\operatorname{depth}_S I'/J' = 1$ by induction hypothesis or by Lemma 1.7. It follows that $\operatorname{depth}_S I/J = 1$ by Lemma 1.5. Thus we may assume that $b_2 = x_2x_3$. Using induction hypothesis and the same argument we may suppose that $b_3 \in (x_1, x_2, x_3)$ and so we may assume $b_3 = x_3x_4$. By recurrence we may assume that $b_k = x_kx_{k+1}$ for $k \in [s-1]$ and $b_s = x_sx_t$ for a certain $t \in [n]$. Note that $t > s$ because otherwise x_r divides no b_k. Thus we may assume that $b_s = x_sx_{s+1}$. It follows that $r = s+1$. Let $S'' = K[x_1, \ldots, x_r]$, $I'' = I \cap S''$, $J'' = J \cap S''$. Then $\operatorname{depth}_{S''} I''/J'' = 1$ by the above case $r = n$. Note that $(x_{r+1}, \ldots, x_n)I \subset J$. Since $I/J \cong (I''S/J''S) \otimes_S S/(x_{r+1}, \ldots, x_n)$ we get $\operatorname{depth}_S I/J = \operatorname{depth}_S(I''S/J''S) - (n-r) = \operatorname{depth}_{S''} I''/J'' = 1$. $\square$

2. Main result

We want to extend Proposition 1.10 for the case $s > 2$. Next examples are illustrations of our method.

Example 2.1. Let $n = 6$, $d = 3$, $f_1 = x_1x_5x_6$, $f_2 = x_2x_4x_6$, $f_3 = x_3x_4x_5$, $f_4 = x_4x_5x_6$, $J = (x_1x_2x_4x_6, x_1x_2x_5x_6, x_1x_3x_4x_5, x_1x_3x_5x_6, x_2x_3x_4x_5, x_2x_3x_4x_6)$ and $I = (f_1, f_2, f_3, f_4)$. We have $s = 3$, $b_1 = x_1f_4 = x_4f_1$, $b_2 = x_2f_4 = x_5f_2$, $b_3 = x_3f_4 = x_6f_3$. Let $S' = K[x_1, \ldots, x_5]$, $f_1' = f_1/x_6$, $f_2' = f_2/x_6$, $f_4' = f_4/x_6$ and $U = (f_1', f_2', f_4')$, $V = (x_1x_2x_4, x_1x_2x_5, x_1x_3x_5, x_2x_3x_4)$ be ideals of S'. In the exact sequence

$$0 \to (I + VS)/VS \to US/VS \to US/(I + VS) \to 0$$

the middle term has the depth ≥ 3 because $\operatorname{depth}_{S'} U/V \geq 2$ by Lemma 1.1. The last term $US/(I + VS)$ has then the depth 2 by Proposition 1.10, since in this case $\mu(U) = 3$ but there exist just two monomials in $US \setminus (I + VS)$ of degree 3, namely $b_1' = x_1x_4x_5 = b_1/x_6$, $b_2' = x_2x_4x_5 = b_2/x_6$ because $b_3/x_6 = f_3 \in I$. By the Depth Lemma it follows that the first term has the depth 3. But the first term is isomorphic with $I/(I \cap VS) = I/J$ since $J = I \cap VS$. Hence $\operatorname{depth}_S I/J = 3$.

Example 2.2. Let $n = 6$, $d = 2$, $f_1 = x_1x_6$, $f_2 = x_1x_5$, $f_3 = x_1x_3$, $f_4 = x_3x_4$, $f_5 = x_2x_4$,

$$J = (x_1x_2x_4, x_1x_2x_5, x_1x_2x_3, x_1x_2x_6, x_1x_3x_6, x_1x_4x_5, x_1x_4x_6,$$

$$x_2x_4x_5, x_2x_4x_6, x_3x_4x_5, x_3x_4x_6)$$

and $I = (f_1, f_2, f_3, f_4, f_5)$. We have $s = 4$, $b_1 = x_5f_1 = x_6f_2$, $b_2 = x_3f_2 = x_5f_3$, $b_3 = x_4f_3 = x_1f_4$, $b_4 = x_2f_4 = x_3f_5$. Let W be the ideal generated by all monomials of degree d which are not divisors of any b_k, $k \in [s]$. Then $W = (x_1x_2, x_2x_5, x_2x_6, x_3x_6, x_4x_5, x_4x_6)$ and note that $I \cap W = J$. Set $T = (x_1, x_4, x_3x_6)$. In the exact sequence

$$0 \to (I + W)/W \to T/W \to T/(I + W) \to 0$$

the middle term has the depth ≥ 2 because $\operatorname{depth}_S S/T = 3$ and $\operatorname{depth}_S S/W \geq 2$ by Lemma 1.1. The last term $T/(I + W)$ has the depth 1 by Corollary 1.9, since in this case there are only 2-generators of degree 1 but there exists just one monomial x_1x_4 in $T \setminus (I + W)$ of degree 2. By the Depth Lemma it follows that the first term has the depth 2. But the first term is isomorphic with $I/(I \cap W) = I/J$. Hence $\operatorname{depth}_S I/J = 2$.

Theorem 2.3. *If $r > s$ then* $\operatorname{depth}_S I/J = d$*, independently of the characteristic of K.*

Proof. Apply induction on s, the case $s \leq 2$ being done in Proposition 1.10. Fix $s > 2$ and apply induction on $d \geq 1$, the case $d = 1$ being done in Proposition 1.11. Using Lemma 1.6 we may suppose that each f_i, $i \in [r]$ divides a certain b_k. Since $r > s$ we may suppose that one b_k is a multiple of two different f_i, let us say $b_1 = x_1f_1 = x_2f_2$. In fact we may assume that each b_k is a multiple of two different f_i because if let us b_s is just a multiple of f_r then we may take

$I' = (f_1, \ldots, f_{r-1})$, $J' = J \cap I'$ and we get $\operatorname{depth}_S I'/J' = 2$ by induction hypothesis on s since $r-1 > s-1$, that is $\operatorname{depth}_S I/J = 2$ by Lemma 1.5.

Set $g = f_1/x_2$, that is g is the greatest common divisor between f_1, f_2. We may suppose that $g|f_i$ if and only if $i \in [e]$ for some $2 \leq e \leq r$. Clearly g has degree $d-1 \leq n-e$ since b_k are square free. If $e = r$ then $I = gI''S$, $J = J''S$ for some monomial ideals I'', J'' of $S'' = K[\{x_i; 1 \leq i \leq n, x_i \not| g\}]$ and $\operatorname{depth}_S I/J = \operatorname{depth}_S I''S/J''S = (d-1) + \operatorname{depth}_{S''} I''/J''$. By the Proposition 1.11 $\operatorname{depth}_{S''} I''/J'' = 1$ and so $\operatorname{depth}_S I/J = d$.

Now we may suppose that $e < r$ and $f_i = x_i g$ for $i \in [e]$. If each f_i, $i > e$ does not divide any b_k, $k \in [e]$ then we may take $I' = (f_{e+1}, \ldots, f_r)$, $J' = J \cap I'$ and we get $\operatorname{depth}_S I'/J' = d$ by induction hypothesis on s since $r-e > s-e$, that is $\operatorname{depth}_S I/J = d$ by Lemma 1.5.

Thus we may suppose that $f_r|b_1$, that is $f_r = x_1x_2g/x_\nu$ for some variable x_ν dividing g. This is because g does not divide f_r. We may assume $\nu = n$ and so $f_r = b_1/x_n = x_1x_2g'$ for $g' = g/x_n$. Let W be the ideal generated by all monomials of S of degree d which are not divisors of any b_k, $k \in [s]$. Set $E = (x_1, \ldots, x_e)g' + I$ and $T = E + W$. In the exact sequence

$$0 \to (I+W)/(J+W) \to T/(J+W) \to T/(I+W) \to 0$$

the middle term is isomorphic with $E/(E \cap (J+W))$ which has the depth $\geq d$ by Lemmas 1.1, 1.2, 1.4 because $E \cap W$ is generated in degree $> d$ and $(x_1, \ldots, x_e)g'$ is generated in the first $(n-1)$-variables.

The last term $T/(I+W)$ has the depth $d-1$ by induction hypothesis on d, since in this case there are e-generators of T of degree $d-1$, but there are at most $(e-1)$-monomials $b_2' = b_2/x_n, \ldots, b_e' = b_e/x_n$ in $T \setminus (I+W)$ of degree d. By the Depth Lemma it follows that the first term has the depth d. The first term is isomorphic with $I/(I \cap (J+W))$. Note that the degree of a square free monomial u from $I \cap W$ is $\geq d+1$ and if it is $d+1$ then it is not a b_k because the generators of W do not divide b_k, that is $u \in J$. Thus $I \cap (J+W)$ and J have the same monomials of degree $d+1$ and so $\operatorname{depth}_S I/(I \cap (J+W)) = \operatorname{depth}_S I/J = d$ by Lemma 1.2. $\square$

The condition given in Theorem 2.3 is tight as shows the following two examples.

Example 2.4. Let $n = 4$, $d = 2$, $f_1 = x_1x_3$, $f_2 = x_2x_4$, $f_3 = x_1x_4$ and $I = (f_1, \ldots, f_3)$, $J = (x_2x_3x_4)$ be ideals of S. We have $r = s = 3$, $b_1 = x_1x_2x_3$, $b_2 = x_1x_2x_4$, $b_3 = x_1x_3x_4$, and $\operatorname{depth}_S I/J = d+1$.

Example 2.5. Let $n = 6$, $d = 2$, $f_1 = x_1x_5$, $f_2 = x_2x_3$, $f_3 = x_3x_4$, $f_4 = x_1x_6$, $f_5 = x_1x_4$, $f_6 = x_1x_2$, and $I = (f_1, \ldots, f_6)$,

$$J = (x_1x_2x_4, x_1x_2x_5, x_1x_3x_5, x_1x_3x_6, x_1x_4x_6, x_2x_3x_5, x_2x_3x_6, x_3x_4x_5, x_3x_4x_6).$$

We have $r = s = 6$ and $b_1 = x_1x_4x_5$, $b_2 = x_2x_3x_4$, $b_3 = x_1x_2x_3$, $b_4 = x_1x_5x_6$, $b_5 = x_1x_3x_4$, $b_6 = x_1x_2x_6$ but $\operatorname{depth}_S I/J = 2$.

Remark 2.6. The above example shows that one could find a nice class of factors of square free monomial ideals with $r = s$ but $\operatorname{depth}_S I/J = d$ similarly as in [7, Lemma 6]. An important tool seems to be a classification of the possible posets given on $f_1, \ldots, f_r, b_1, \ldots, b_s$ by the divisibility.

Remark 2.7. Given $J \subset I$ two square free monomial ideals of S as above one can consider the poset $P_{I\setminus J}$ of all square free monomials of $I \setminus J$ (a finite set) with the order given by the divisibility. Let $\mathcal{P}$ be a partition of $P_{I\setminus J}$ in intervals $[u, v] = \{w \in P_{I\setminus J} : u|w, w|v\}$, let us say $P_{I\setminus J} = \cup_i [u_i, v_i]$, the union being disjoint. Define $\operatorname{sdepth} \mathcal{P} = \min_i \deg v_i$ and $\operatorname{sdepth}_S I/J = \max_{\mathcal{P}} \operatorname{sdepth} \mathcal{P}$, where $\mathcal{P}$ runs in the set of all partitions of $P_{I\setminus J}$. This is the so called the Stanley depth of I/J, in fact this is an equivalent definition given in a general form in [9], [1]. If $r > s$ then it is obvious that $\operatorname{sdepth}_S I/J = d$ and so Theorem 2.3 says that Stanley's Conjecture holds, that is $\operatorname{sdepth}_S I/J \geq \operatorname{depth}_S I/J$. In general the Stanley depth of a monomial ideal I is greater or equal with the Lyubeznik' size of I increased by one (see [2]). Stanley's Conjecture holds for intersections of four monomial prime ideals of S by [4] and [6] and for square free monomial ideals of $K[x_1, \ldots, x_5]$ by [5] (a short exposition on this subject is given in [7]). Also Stanley's Conjecture holds for intersections of three monomial primary ideals by [10]. If I is generated by r-square free monomials of degree d then $\operatorname{sdepth}_S I = d$ if and only if $r > \binom{n}{d+1}$ as shows [8, Corollary 10] extending [3, Corollary 2.2]. A similar result for factors of square free monomial ideals is still not done, though should hold.

3. Around Theorem 2.3

Let $S' = K[x_1, \ldots, x_{n-1}]$ be a polynomial ring in $n - 1$ variables over a field K, $S = S'[x_n]$ and $U, V \subset S'$, $V \subset U$ be two square free monomial ideals. Set $W = (V + x_n U)S$. Actually, every monomial square free ideal T of S has this form because then $(T : x_n)$ is generated by an ideal $U \subset S'$ and $T = (V + x_n U)S$ for $V = T \cap S'$.

Lemma 3.1. *([5]) Suppose that $U \neq V$ and $\operatorname{depth}_{S'} S'/U = \operatorname{depth}_{S'} S'/V = \operatorname{depth}_{S'} U/V$. Then $\operatorname{depth}_S S/W = \operatorname{depth}_{S'} S'/U$.*

Lemma 3.2. *Suppose that $U \neq V$ and $d := \operatorname{depth}_{S'} S'/U = \operatorname{depth}_{S'} S'/V$. Then $d = \operatorname{depth}_{S'} U/V$ if and only if $d = \operatorname{depth}_S S/W$.*

Proof. The necessity follows from the above lemma. For sufficiency note that in the exact sequence

$$0 \to VS \to W \to US/VS \to 0$$

the depth of the left end is $d + 2$ and the middle term has depth $d + 1$. It follows that $\operatorname{depth}_S US/VS = d + 1$ by the Depth Lemma, which is enough. $\square$

Let I be an ideal of S generated by square free monomials of degree $\geq d$ and $x_n f_1, \ldots, x_n f_r$, $r > 0$ be the square free monomials of $I \cap (x_n)$ of degree d. Set $U = (f_1, \ldots, f_r)$, $V = I \cap S'$.

Theorem 3.3. *If $r > \rho_d(U) - \rho_d(U \cap V)$ then $\operatorname{depth}_S S/I = \operatorname{depth}_{S'}(U + V)/V = d - 1$.*

Proof. By Theorem 2.3 we have $\operatorname{depth}_{S'}(U + V)/V = \operatorname{depth}_{S'} U/(U \cap V) = d - 1$. Using Lemmas 1.2, 1.4 we get

$$\operatorname{depth}_{S'}(U + V)/V = \operatorname{depth}_{S'}((I : x_n) \cap S')/(I \cap S') = d - 1.$$

If $\operatorname{depth}_{S'} S'/(I \cap S') = \operatorname{depth}_{S'} S'/((I : x_n) \cap S') = d-1$ then $\operatorname{depth}_S S/I = d-1$ by Lemma 3.2. If $\operatorname{depth}_{S'} S'/((I : x_n) \cap S') = d-2$ then in the exact sequence

$$0 \to S/(I : x_n) \xrightarrow{x_n} S/I \to S'/(I \cap S') \to 0$$

the first term has depth $d-1$ and the other two have depth $\geq d-1$ by Lemma 1.1. By the Depth Lemma it follows that $\operatorname{depth}_S S/I = d-1$.

It remains to consider the case when at least one from $\operatorname{depth}_{S'} S'/((I : x_n) \cap S')$, $\operatorname{depth}_{S'} S'/(I \cap S')$ is $\geq d$. Using the Depth Lemma in the exact sequence

$$0 \to ((I : x_n) \cap S')/(I \cap S') \to S'/(I \cap S') \to S'/((I : x_n) \cap S') \to 0$$

we see that necessarily the depth of the last term is $\geq d$ and the depth of the middle term is $d-1$. But then the Depth Lemma applied to the previous exact sequence gives $\operatorname{depth}_S S/I = d-1$ too. $\square$

The following corollary extends [8, Corollary 3].

Corollary 3.4. *Let I be an ideal generated by $\mu(I) > 0$ square free monomials of degree d. If $\mu(I) \geq \rho_{d+1}(I)$, in particular if $\mu(I) \geq \binom{n}{d+1}$, then* $\operatorname{depth}_S I = d$.

Proof. We have $I = (V + x_n(U+V))S$ as above. Renumbering the variables we may suppose that $V \neq 0$. Note that $\mu(I) = r + \rho_d(V)$ and $\rho_{d+1}(I) = \rho_{d+1}(V) + \rho_d(U+V) > \rho_d(V) + \rho_d(U) - \rho_d(U \cap V)$. By hypothesis, $\mu(I) \geq \rho_{d+1}(I)$ and so $r > \rho_d(U) - \rho_d(U \cap V)$. Applying Theorem 3.3 we get $\operatorname{depth}_S S/I = d-1$, which is enough. $\square$

Remark 3.5. Take in Example 2.4 $S' = K[x_1, \ldots, x_5]$ and $L = (J + x_5 I)S'$. We have $\mu(L) = 4 < \binom{5}{3+1}$, that is the hypothesis of the above corollary are not fulfilled. This is the reason that $\operatorname{depth}_{S'} L \geq 3$ by Lemma 3.2 since $\operatorname{depth}_S I/J = 3$. Thus the condition of the above corollary is tight.

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